

Variational solutions of stochastic partial differential equations with cylindrical Lévy noise

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Abstract

In this article, the existence of a unique solution in the variational approach of the stochastic evolution equation

$$dX(t) = F(X(t)) dt + G(X(t)) dL(t)$$

driven by a cylindrical Lévy process L is established. The coefficients F and G are assumed to satisfy the usual monotonicity and coercivity conditions. The noise is modelled by a cylindrical Lévy processes which is assumed to belong to a certain subclass of cylindrical Lévy processes and may not have finite moments.

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1 Introduction

Cylindrical Brownian motion, or equivalently Gaussian space-time noise, has been the standard model for the driving noise for stochastic partial differential equations

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(SPDEs) for the last 50 years. Cylindrical Brownian motions are naturally extended to the class of cylindrical Lévy processes in order to capture non-Gaussian and discontinuous random perturbations; see Applebaum and Riedle [2]. However, in contrast to genuine Lévy processes, cylindrical Lévy processes do not enjoy a semimartingale decomposition and do not attain values in the underlying space. Consequently, conventional methods, such as martingale arguments or stopping time techniques, are no longer applicable in the general case.

If the driving noise is only additive, various specific examples of cylindrical Lévy processes as driving noise of (linear) SPDEs are considered; see e.g. Brzeźniak and Zabczyk [6], Peszat and Zabczyk [22] and Priola and Zabczyk [25]. A general framework of linear equations with additive noise modelled by arbitrary cylindrical Lévy processes is developed in Riedle [28] and Kumar and Riedle [16] and [17]. The case of an SPDE with a multiplicative perturbation is only considered in Riedle [27], however under the restrictive assumption of weak square-integrability of the driving cylindrical Lévy process. In this case, similar methods as for cylindrical Brownian motions can be exploited.

It is the first time in this work, that an SPDE with multiplicative noise is investigated, where the random perturbation is modelled by a cylindrical Lévy process without assuming any conditions on the moments. The existence of a solution in this situation cannot be necessarily anticipated from the square-integrable case or other results: it is known that the irregular jumps of a cylindrical Lévy process, in particular in the case without moments, can cause completely novel phenomena; see e.g. Brzeźniak et al [4] and Priola and Zabczyk [25]. We focus on a subclass of cylindrical Lévy processes which are extensively investigated in the literature as driving noise of additive equations; see e.g. [4, 19, 22, 24, 25].

More specifically, in this paper we consider an evolution equation of the form

$$dX(t) = F(X(t)) dt + G(X(t)) dL(t), \quad (1.1)$$

in the variational approach where L is a cylindrical Lévy process. The coefficients F and G are given operators and are assumed to satisfy standard assumptions such as monotonicity and coercivity. The variational approach, first in a deterministic and then in a stochastic setting, goes back to the works by Bensoussan, Lions and Pardoux; a brief history of the approach can be found in [15]. Existence results for equations of the form (1.1) but driven by a Brownian motion were derived in Krylov and Rozovskii [15]. In a series of publications [9, 10, 11], Gyöngy and Krylov generalised these results to semimartingales as driving noises. The variational approach has been extended in many directions. It is especially worth to mention the works of Liu and Röckner [18] and Prévôt and Röckner [23], where the assumptions on the coefficients were relaxed, such that classical models for example from fluid dynamics are captured by the framework. Recently, Brzeźniak, Liu and Zhu [5] have considered equations of the form (1.1)

with locally monotone coefficients and driven by a Lévy-type noise, i.e. a noise which originates from a generalisation of the Gaussian space-time white noise.

The driving noise L in (1.1) is assumed to be a member of a certain subclass of cylindrical Lévy processes. Elements of this subclass may not have finite moments but they enjoy a certain kind of semimartingale decomposition. Nevertheless, classical stopping time arguments or other standard techniques cannot be exploited since the unbounded potential of the cylindrical Lévy process causes divergence of the Galerkin approximation. We circumvent this problem by introducing weights in each dimension accordingly. We present several examples of cylindrical Lévy processes of the subclass under consideration, and relate those to models considered in the literature.

The organisation of the paper is as follows. In Section 2, we collect some preliminaries and give the definition of cylindrical Lévy processes. In Section 3, we establish the existence of a solution of equation (1.1) in the case of a cylindrical Lévy process with weak second moments. Since the arguments are similar to the classical case, investigated in Brzeźniak, Liu and Zhu [5], we omit details, which can be found in Kosmala [14]. In Section 4, we introduce the subclass of cylindrical Lévy processes under consideration, and define their stochastic integrals. Section 5 is devoted to establishing the existence of a unique solution of (1.1) driven by a cylindrical Lévy process from this subclass.

2 Preliminaries

Let U and H be separable Hilbert spaces with norm $\|\cdot\|_U$ and inner product $\langle \cdot, \cdot \rangle_U$ and analogously for H . We fix an orthonormal basis (e_j) of U and (f_j) of H . The Borel σ -algebra is denoted by $\mathfrak{B}(U)$. We use $L(U, H)$ to denote the space of bounded operators from U to H equipped with the operator norm. The subspace of Hilbert-Schmidt operators from U to H is denoted by $L_{\text{HS}}(U, H)$ and it is equipped with the norm

$$\|\varphi\|_{L_{\text{HS}}(U, H)}^2 := \sum_{j=1}^{\infty} \|\varphi e_j\|_H^2.$$

Let (S, \mathcal{S}, μ) be a σ -finite measure space. We denote by $L^p(S; U)$ the Bochner space of all equivalence classes of measurable functions $f: S \rightarrow U$ which are p -th integrable with respect to μ equipped with the usual norm. We use $L^0(S; U)$ to denote the space of all equivalence classes of measurable functions $f: S \rightarrow U$ with the topology induced by convergence in measure. The underlying measure μ and the σ -algebra \mathcal{S} are always obvious from the context, e.g. if $S = [0, T] \times \Omega$ then $\mu = dt \otimes P$ and $\mathcal{S} = \mathfrak{B}([0, T]) \otimes \mathcal{F}$, where dt is the Lebesgue measure on $[0, T]$ and (Ω, \mathcal{F}, P) is a probability space.

For a subset Γ of U , sets of the form

$$C(u_1, \dots, u_n; B) := \{u \in U : (\langle u, u_1 \rangle, \dots, \langle u, u_n \rangle) \in B\},$$

for $u_1, \dots, u_n \in \Gamma$ and $B \in \mathfrak{B}(\mathbb{R}^n)$ are called *cylindrical sets with respect to Γ* ; the set of all these cylindrical sets is denoted by $\mathcal{Z}(U, \Gamma)$ and it is an algebra. If Γ is finite then it is a σ -algebra. A function $\lambda: \mathcal{Z}(U, U) \rightarrow [0, \infty]$ is called a *cylindrical measure*, if for each finite subset $\Gamma \subseteq U$ the restriction of λ on the σ -algebra $\mathcal{Z}(U, \Gamma)$ is a measure. A cylindrical measure is called a cylindrical probability measure if $\lambda(U) = 1$. A *cylindrical random variable* Z in U is a linear and continuous map $Z: U \rightarrow L^0(\Omega; \mathbb{R})$. Each cylindrical random variable Z defines a cylindrical probability measure λ by

$$\lambda: \mathcal{Z}(U, U) \rightarrow [0, 1], \quad \lambda(C) = P((Zu_1, \dots, Zu_n) \in B),$$

for cylindrical sets $C = C(u_1, \dots, u_n; B)$. The cylindrical probability measure λ is called the *cylindrical distribution* of Z . The characteristic function of a cylindrical random variable Z is defined by

$$\varphi_Z: U \rightarrow \mathbb{C}, \quad \varphi_Z(u) = \mathbb{E}[\exp(iZu)],$$

and it uniquely determines the cylindrical distribution of Z .

Let $(\mathcal{F}_t: t \geq 0)$ be a filtration satisfying the usual conditions. A family of cylindrical random variables $L(t): U \rightarrow L^0(\Omega; \mathbb{R})$, $t \geq 0$, is called a *cylindrical Lévy process* if for any $n \in \mathbb{N}$ and $u_1, \dots, u_n \in U$ we have that $((L(t)u_1, \dots, L(t)u_n): t \geq 0)$ is a Lévy process in \mathbb{R}^n with respect to the filtration (\mathcal{F}_t) . A version of this definition appeared for the first time in Applebaum and Riedle [2] with further modifications in [27]. Here, we include a filtration in the definition. The characteristic function of $L(t)$ can be written in the form

$$\varphi_{L(1)}(u) = \exp \left(ip(u) - \frac{1}{2}q(u) + \int_U \left(e^{i\langle u, x \rangle} - 1 - i\langle u, x \rangle \mathbb{1}_{B_{\mathbb{R}}}(\langle u, x \rangle) \right) \nu(dx) \right); \quad (2.1)$$

see [2, Th. 2.7] or [26, Th. 3.4]. In the above formula, $B_{\mathbb{R}}$ is the closed unit ball in \mathbb{R} , $p: U \rightarrow \mathbb{R}$ is a continuous function with $p(0) = 0$, $q: U \rightarrow \mathbb{R}$ is a quadratic form, and ν is a cylindrical measure on $\mathcal{Z}(U, U)$ satisfying

$$\int_U (\langle u, v \rangle^2 \wedge 1) \nu(dv) < \infty \quad \text{for all } u \in U. \quad (2.2)$$

A cylindrical measure satisfying (2.2) is called a *cylindrical Lévy measure*.

We say that L is *weakly square-integrable* or that it has *weak second moments* if $\mathbb{E}[|L(t)u|^2] < \infty$ for all $t \geq 0$ and $u \in U$. In this case, it follows from the closed graph theorem that $L(t): U \rightarrow L^2(\Omega; \mathbb{R})$ is continuous for each $t \geq 0$. Similarly L is said to be *weakly mean-zero* if $\mathbb{E}[L(t)u] = 0$ for $t \geq 0$ and $u \in U$. For a weakly mean-zero cylindrical Lévy process L , the *covariance operator* $Q: U \rightarrow U$ is a non-negative and symmetric linear operator defined by $\langle Qu, v \rangle = \mathbb{E}[L(1)uL(1)v]$ for each $u, v \in U$.

3 Case of finite second moments

In this section L is assumed to have weak second moments. Let $Q: U \rightarrow U$ be its covariance operator. We improve the theory of stochastic integration with respect to a weakly square-integrable cylindrical Lévy processes introduced in [27] by extending the space of integrands so that it depends on the characteristics of the integrator. Let $\mathcal{H} = Q^{1/2}U$ be equipped with the scalar product $\langle u, v \rangle_{\mathcal{H}} = \langle Q^{-1/2}u, Q^{-1/2}v \rangle_U$ for $u, v \in \mathcal{H}$. Then $(Q^{1/2}e_n)$ is an orthonormal basis of \mathcal{H} and $\psi \in L_{\text{HS}}(\mathcal{H}, H)$ if and only if $\psi Q^{1/2}$ in $L_{\text{HS}}(U, H)$ and their norms coincide; see [23, Sec. 2.3.2].

The approach is based on the observation that the cylindrical increments of a cylindrical Lévy process can be Radonified by a *random* Hilbert-Schmidt mapping. More specifically, for $0 \leq s \leq t$ let $\Phi: \Omega \rightarrow L_{\text{HS}}(\mathcal{H}, H)$ be a simple, \mathcal{F}_s -measurable random variable of the form

$$\Phi(\omega) = \sum_{j=1}^m \mathbb{1}_{A_j}(\omega) \varphi_j$$

for deterministic operators $\varphi_j \in L_{\text{HS}}(\mathcal{H}, H)$ and sets $A_j \in \mathcal{F}_s$ for $j = 1, \dots, m$. The Hilbert-Schmidt property implies that for each $j \in \{1, \dots, m\}$ there exists a *genuine* random variable $J_{s,t}\varphi_j: \Omega \rightarrow H$ such that $(L(t) - L(s))(\varphi_j^* h) = \langle J_{s,t}\varphi_j, h \rangle$ for all $h \in H$. By linearity one can define a random variable $J_{s,t}\Phi: \Omega \rightarrow H$ satisfying

$$\langle J_{s,t}\Phi, h \rangle = \sum_{j=1}^m \mathbb{1}_{A_j}(L(t) - L(s))(\varphi_j^* h) \quad \text{for all } h \in H. \quad (3.1)$$

By beginning with simple stochastic processes $(\Psi(t) : t \in [0, T])$ of the form

$$\Psi(t) = \Phi_0 \mathbb{1}_{\{0\}}(t) + \sum_{k=1}^{N-1} \Phi_k \mathbb{1}_{(t_k, t_{k+1}]}(t) \quad \text{for } t \in [0, T],$$

where $0 = t_1 < t_2 < \dots < t_N = T$ and each Φ_k is a simple \mathcal{F}_{t_k} -measurable $L_{\text{HS}}(U, H)$ -valued random variable, one can define the stochastic integral as

$$\int_0^T \Psi(s) dL(s) = \sum_{k=1}^{N-1} J_{t_k, t_{k+1}} \Phi_k.$$

Using the Itô isometry (see [27])

$$\mathbb{E} \left[\left\| \int_0^T \Psi(s) dL(s) \right\|_H^2 \right] = \int_0^T \mathbb{E} \left[\|\Psi(s) Q^{1/2}\|_{L_{\text{HS}}(U, H)}^2 \right] ds$$

one can extend the definition to all stochastic processes in the space

$$\Lambda := \left\{ \Psi : [0, T] \times \Omega \rightarrow L_{\text{HS}}(\mathcal{H}, H) : \text{predictable and} \right. \\ \left. \mathbb{E} \left[\int_0^T \|\Psi(t)Q^{1/2}\|_{L_{\text{HS}}(U, H)}^2 dt \right] < \infty \right\}.$$

The space Λ becomes a Banach space with the norm defined by

$$\|\Psi\|_{\Lambda} = \left(\mathbb{E} \left[\int_0^T \|\Psi(t)Q^{1/2}\|_{L_{\text{HS}}(U, H)}^2 dt \right] \right)^{1/2}, \quad \Psi \in \Lambda.$$

In the following theorem we summarise the properties of this integral. If L is a genuine Lévy process, formulas for the angle bracket processes are well known; see [21, Cor. 8.17]. Recall that for a square-integrable, H -valued martingale M , the angle bracket process $\langle M, M \rangle$ is defined as the unique increasing, predictable process such that $(\|M(t)\|^2 - \langle M, M \rangle(t) : t \geq 0)$ is a martingale.

Theorem 3.1. *Suppose that L is weakly square-integrable cylindrical Lévy process and $\Psi \in \Lambda$.*

(i) *The process*

$$\left(I(t) = \int_0^t \Psi(s) dL(s) : t \in [0, T] \right)$$

is a square integrable martingale and has a modification with càdlàg trajectories.

$$(ii) \quad \langle I(\Psi), I(\Psi) \rangle(t) = \int_0^t \|\Psi(s)Q^{1/2}\|_{L_{\text{HS}}(U, H)}^2 ds \quad \text{for all } t \in [0, T] \text{ } P\text{-a.s.}$$

(iii) *For any stopping time τ with $P(\tau \leq T) = 1$ we have*

$$\int_0^{t \wedge \tau} \Psi(s) dL(s) = \int_0^t \Psi(s) \mathbb{1}_{\{s \leq \tau\}} dL(s) \quad \text{for all } t \in [0, T] \text{ } P\text{-a.s.}$$

Proof. Similar results are discussed for integrals with respect to classical Lévy processes, for (i) see [21, Th. 8.7], for (ii) see [21, Cor. 8.17] and for (iii) see [23, Lem. 2.3.9]. The derivations for the cylindrical case using the properties of the Radonified increments such as (3.1) are straightforward and can be found in [14]. \square

We now discuss the existence and uniqueness of solution in the square-integrable case. Let $(V, \|\cdot\|_V)$ be a separable reflexive Banach space and let $(H, \langle \cdot, \cdot \rangle_H)$ and $(U, \langle \cdot, \cdot \rangle_U)$ be separable Hilbert spaces. Let V^* and H^* denote their duals. Assume that V is densely and continuously embedded into H . That is we have a Gelfand triple

$$V \subseteq H = H^* \subseteq V^*.$$

Further, denote with ${}_{V^*}\langle \cdot, \cdot \rangle_V$ the duality pairing of V^* and V . For all $h \in H$ and $v \in V$ we have ${}_{V^*}\langle h, v \rangle_V = \langle h, v \rangle_H$ and without loss of generality we may assume that $\|v\|_H \leq \|v\|_V$ for $v \in V$ and $\|h\|_{V^*} \leq \|h\|_H$ for $h \in H$.

We consider the equation

$$dX(t) = F(X(t)) dt + G(X(t)) dL(t) \quad \text{for } t \in [0, T], \quad (3.2)$$

with the initial condition $X(0) = X_0$ for a square-integrable, \mathcal{F}_0 -measurable random variable X_0 . The driving noise is a cylindrical Lévy process on a separable Hilbert space U . In this section we assume that L is a weakly mean-zero, weakly square-integrable, cylindrical Lévy process, i.e. a cylindrical martingale with covariance operator $Q: U \rightarrow U$. In the remainder of the paper, we assume that there exists an orthonormal basis (e_n) of U consisting of eigenvectors of Q . Note that this does not follow from the conditions on Q , since it needs not be compact. The coefficients in equation (3.2) are given by functions $F: V \rightarrow V^*$ and $G: V \rightarrow L_{\text{HS}}(\mathcal{H}, H)$. More specifically, we assume the following in this section: there are constants $\alpha, \lambda, \beta, c > 0$ such that:

(A1) (Coercivity) for all $v \in V$ we have

$$2{}_{V^*}\langle F(v), v \rangle_V + \|G(v)Q^{1/2}\|_{L_{\text{HS}}(U, H)}^2 + \alpha\|v\|_V^2 \leq \lambda\|v\|_H^2 + \beta;$$

(A2) (Monotonicity) for all $v_1, v_2 \in V$, we have

$$2{}_{V^*}\langle F(v_1) - F(v_2), v_1 - v_2 \rangle_V + \|(G(v_1) - G(v_2))Q^{1/2}\|_{L_{\text{HS}}(U, H)}^2 \leq \lambda\|v_1 - v_2\|_H^2;$$

(A3) (Linear growth) $\|F(v)\|_{V^*} \leq c(1 + \|v\|_V)$ for all $v \in V$;

(A4) (Hemicontinuity) the mapping $\mathbb{R} \ni s \mapsto {}_{V^*}\langle F(v_1 + sv_2), v_3 \rangle_V$ is continuous for all $v_1, v_2, v_3 \in V$.

(A5) The cylindrical Lévy process L is weakly mean-zero and is weakly square-integrable. Its covariance operator Q has eigenvectors (e_j) , which form an orthonormal basis of U .

Conditions of this form appear in most of the papers mentioned in the introduction. For instance in [5] these conditions are formulated for a sum of cylindrical Brownian motion with covariance equal to the identity (in particular, satisfying (A5)) and a Poisson random measure. Later, we will consider the case of a non-integrable noise. We will obtain the existence and uniqueness of solutions by truncating the jumps and reducing the problem to the case of an equation driven by a process satisfying (A5). We now give the definition of a solution to (3.2), similarly as in Prévôt and Röckner [23, Def. 4.2.1] or Brzeźniak, Liu and Zhu [5, Def. 1.1]. Since we later consider the case of a driving noise without finite moments and thus the solution cannot be expected to have finite moments, we do not require finite expectation of the solution.

Definition 3.2. A *variational solution* of (3.2) is a pair (X, \bar{X}) of an H -valued, càdlàg adapted process X and a V -valued, predictable process \bar{X} such that

- (i) X equals \bar{X} $dt \otimes P$ -almost everywhere;
- (ii) P -a.s. $\int_0^T \|\bar{X}(t)\|_V dt < \infty$;
- (iii) $X(t) = X_0 + \int_0^t F(\bar{X}(s)) ds + \int_0^t G(\bar{X}(s)) dL(s)$ for all $t \in [0, T]$ P -a.s. (3.3)

We say that the solution is *pathwise unique* if any two variational solutions (X, \bar{X}) and (Y, \bar{Y}) satisfy

$$P(X(t) = Y(t) \text{ for all } t \in [0, T]) = 1.$$

Theorem 3.3. Under Assumptions (A1)–(A5), equation (3.2) has a unique variational solution (X, \bar{X}) . Moreover, the solution satisfies

$$\int_0^T \mathbb{E} \left[\|\bar{X}(s)\|_V^2 \right] ds < \infty.$$

The proof can be obtained by repeating the arguments from [23] and using the properties of the integral outlined in Theorem 3.1. The details are presented in [14].

4 Orthogonal cylindrical Lévy processes

In this section we consider the case of a driving noise without finite moments. Contrary to the classical case of a genuine Lévy process, one cannot directly apply stopping time arguments such as in [21, Sec. 9.7] or interlacing techniques such as in [12, Th. IV.9.1], since the cylindrical Lévy process does not attain values in the underlying space.

For a bounded sequence of positive real numbers $c = (c_j) \in \ell^\infty(\mathbb{R}_+)$ we define the sequence of stopping times by

$$\tau_n^c(k) := \inf \left\{ t \geq 0 : \sum_{j=1}^n (\Delta L(t)e_j)^2 c_j^2 > k^2 \right\} \quad \text{for each } k > 0, n \in \mathbb{N}.$$

The stopping time $\tau_n^c(k)$ can be seen as the first time, the n -dimensional Lévy process $((L(t)(c_1 e_1), \dots, L(t)(c_n e_n)) : t \geq 0)$ has a jump of size larger than k . Since $\tau_n^c(k)$ is non-increasing in n , we can define another sequence of stopping times by

$$\tau^c(k) := \lim_{n \rightarrow \infty} \tau_n^c(k) \quad \text{for } k > 0. \tag{4.1}$$

Contrary to the the case of a genuine Hilbert space-valued Lévy process, if the noise is cylindrical the stopping times $\tau_n^c(k)$ may accumulate at zero, i.e. $\tau^c(k) = 0$ P -almost

surely, see Remark 4.6 below. It will turn out that the distribution of the stopping time $\tau^c(k)$ depends on the parameter

$$m^c(k) := \sup_{n \in \mathbb{N}} \nu \left(\left\{ u \in U : \sum_{j=1}^n \langle u, e_j \rangle^2 c_j^2 > k^2 \right\} \right) \quad \text{for } k > 0, \quad (4.2)$$

where ν is the cylindrical Lévy measure of L . If L is a genuine Lévy process in U then its Lévy measure ν is finite outside each ball around 0 and $m^c(k) \rightarrow 0$ as $k \rightarrow \infty$. In the cylindrical case, the situation turns out to be rather different as Proposition 4.1 shows:

Proposition 4.1. Let L be a cylindrical Lévy process with $\tau^c(k)$ defined in (4.1) and m^c defined in (4.2) for a fixed $c \in \ell^\infty(\mathbb{R}_+)$.

(1) We have the following dichotomy for each $k > 0$:

- (i) $m^c(k) = 0 \Leftrightarrow \tau^c(k) = \infty$ P -a.s;
- (ii) $m^c(k) \in (0, \infty) \Leftrightarrow \tau^c(k)$ is exponentially distributed with parameter $m^c(k)$;
- (iii) $m^c(k) = \infty \Leftrightarrow \tau^c(k) = 0$ P -a.s.

(2) We have: $\lim_{k \rightarrow \infty} m^c(k) = 0 \Leftrightarrow \lim_{k \rightarrow \infty} \tau^c(k) = \infty$ P -a.s.

Proof. (1) Define the mapping

$$\pi_n^c : U \rightarrow U, \quad \pi_n^c(u) = \sum_{j=1}^n c_j \langle u, e_j \rangle e_j.$$

Note that $\tau_n^c(k)$ is the time of the first jump of size larger than k of the finite dimensional Lévy process L_n^c defined by

$$L_n^c(t) = \sum_{j=1}^n c_j L(t)(e_j) e_j, \quad t \geq 0.$$

As the Lévy measure ν_n^c of L_n^c is given by $\nu_n^c := \nu \circ (\pi_n^c)^{-1}$, the stopping time $\tau_n^c(k)$ is exponentially distributed with parameter

$$\lambda_n^k := \nu_n^c(\{u \in U : \|u\|_U > k\}) = \nu \left(\left\{ u \in U : \sum_{j=1}^n c_j^2 \langle u, e_j \rangle^2 > k^2 \right\} \right).$$

(i): the very definition implies that $m^c(k) = 0$ if and only if $\lambda_n^k = 0$ for all $n \in \mathbb{N}$. The latter is equivalent to $\tau_n^c(k) = \infty$ for all $n \in \mathbb{N}$.

(ii), (iii): the characteristic function $\varphi_{\tau_n^c(k)}$ of $\tau_n^c(k)$ is given by

$$\varphi_{\tau_n^c(k)}: \mathbb{R} \rightarrow \mathbb{C}, \quad \varphi_{\tau_n^c(k)}(x) = \frac{\lambda_n^k}{\lambda_n^k - ix}.$$

As λ_n^k monotonically increases to $m^c(k)$ as $n \rightarrow \infty$, the characteristic function $\varphi_{\tau_n^c(k)}$ converges to the characteristic function either of the exponential distribution with parameter $m^c(k)$ or of the Dirac measure in 0.

For establishing (2), note that monotonicity of $k \mapsto \tau^c(k)$ yields

$$P\left(\lim_{k \rightarrow \infty} \tau^c(k) = \infty\right) = P\left(\bigcap_{t \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} \{\tau^c(k) > t\}\right) = \lim_{t \rightarrow \infty} \lim_{n \rightarrow \infty} P(\tau^c(n) > t).$$

Since $P(\tau^c(k) > t) = \exp(-tm^c(k))$, this completes the proof of (2). \square

We now focus on a special class of cylindrical Lévy processes, which, similarly as in the case of a cylindrical Brownian motion, can be represented by a sum. That is we assume a form of the noise as in the Karhunen-Loève theorem with independent components but without requiring finite second moments. Let L be a cylindrical Lévy process with cylindrical Lévy measure ν and let (e_j) be an orthonormal basis of U . L is called *orthogonal* cylindrical Lévy processes if it is of the form

$$L(t)u = \sum_{j=1}^{\infty} \ell_j(t) \langle u, e_j \rangle \quad P\text{-a.s. for all } u \in U, t \geq 0, \quad (4.3)$$

where (ℓ_j) is a sequence of independent, not necessarily identically distributed, one-dimensional Lévy processes. Denote the characteristics (with respect to the standard truncation function $\mathbb{1}_{B_{\mathbb{R}}}$) of ℓ_j by (b_j, s_j, ρ_j) for each $j \in \mathbb{N}$. Lemma 4.2 in [28] guarantees that the sum in (4.3) converges and defines a cylindrical Lévy process if and only if the characteristic functions of ℓ_j are equicontinuous at 0 and the following three conditions are satisfied for every $(\alpha_j) \in \ell^2(\mathbb{R})$:

$$(i) \quad \sum_{j=1}^{\infty} \mathbb{1}_{B_{\mathbb{R}}}(\alpha_j) |\alpha_j| \left| b_j + \int_{1 < |x| \leq |\alpha_j|^{-1}} x \rho_j(dx) \right| < \infty, \quad (4.4)$$

$$(ii) \quad (s_j) \in \ell^{\infty}(\mathbb{R}), \quad (4.5)$$

$$(iii) \quad \sum_{j=1}^{\infty} \int_{\mathbb{R}} (|\alpha_j x|^2 \wedge 1) \rho_j(dx) < \infty. \quad (4.6)$$

Independence of the Lévy processes (ℓ_j) implies that the cylindrical Lévy measure of L has only support in $\cup \text{Span}\{e_j\}$. Consequently, the function m^c , defined in (4.2), reduces to

$$m^c(k) = \sum_{j=1}^{\infty} \rho_j \left(\left\{ x \in \mathbb{R} : |x| > \frac{k}{c_j} \right\} \right) \quad \text{for all } k > 0. \quad (4.7)$$

In general, due to the non-linearity of the truncation function, cylindrical Lévy processes do not enjoy a type of Lévy-Itô decomposition. However, the specific construction of cylindrical Lévy processes of the form (4.3) suggests to derive a Lévy-Itô decomposition from an appropriate decomposition of the real-valued processes ℓ_j . More precisely, for a given sequence $c = (c_j) \in \ell^\infty(\mathbb{R}_+)$ and $k > 0$ we obtain $\ell_j(t) = p_j^{c,k}(t) + m_j^{c,k}(t) + r_j^{c,k}(t)$ for all $t \geq 0$ where

$$p_j^{c,k}(t) := \left(b_j + \int_{1 < |x| \leq k/c_j} x \rho_j(dx) \right) t, \quad (4.8)$$

$$m_j^{c,k}(t) := \sqrt{s_j} W_j(t) + \int_{|x| \leq k/c_j} x \tilde{N}_j(t, dx), \quad (4.9)$$

$$r_j^{c,k}(t) := \int_{|x| > k/c_j} x N_j(t, dx). \quad (4.10)$$

Here, the process W_j is a real-valued standard Brownian motion and N_j is a Poisson random measure on $[0, \infty) \times \mathbb{R}$ with intensity measure $dt \otimes \rho_j$.

In the following lemma, we summarise the conditions on the cylindrical Lévy process such that the stopping times $\tau^c(k)$, defined in (4.1), do not accumulate at zero and such that the decomposition of ℓ_j leads to a decomposition of the cylindrical Lévy process:

(A6) there exists a sequence $c = (c_j) \in \ell^\infty(\mathbb{R}_+)$ such that

$$(i) \quad \left(p_j^{c,k}(1) \right)_{j \in \mathbb{N}} \in \ell^2(\mathbb{R}) \text{ for each } k > 0; \quad (4.11)$$

$$(ii) \quad \sup_{j \in \mathbb{N}} \int_{|x| \leq k/c_j} x^2 \rho_j(dx) < \infty \text{ for each } k > 0; \quad (4.12)$$

$$(iii) \quad \lim_{k \rightarrow \infty} m^c(k) = 0. \quad (4.13)$$

Remark 4.2. Assume that L is of the form (4.3), i.e. Conditions (4.4)–(4.6) are satisfied. For a square summable sequence (c_j) , condition (4.6) implies (4.13) by (4.7) and Lebesgue's theorem. On the other hand, if c_j is constantly equal to 1, then Condition (4.6) implies (4.12). Indeed, suppose for contradiction that the sequence

$(\int_{|x| \leq k} x^2 \rho_j(dx) : j \in \mathbb{N})$ is unbounded. Then there exists a sequence $(\alpha_j) \in \ell^2(\mathbb{R})$ such that

$$\sum_{j=1}^{\infty} \alpha_j^2 \int_{|x| \leq k} x^2 \rho_j(dx) = \infty,$$

which contradicts (4.6).

In summary, for the assumption (A6) to hold there must be some balance between the rate of decay of the Lévy measures (ρ_j) and the rate of convergence of the sequence c .

Lemma 4.3. Assume that L is a cylindrical Lévy process of the form (4.3) satisfying (A6) for a sequence $c \in \ell^\infty(\mathbb{R}_+)$. Then L can be decomposed into $L(t) = P_k^c(t) + M_k^c(t) + R_k^c(t)$ for each $t \geq 0$ and $k > 0$, where P_k^c , M_k^c and R_k^c are cylindrical Lévy processes defined by

$$\begin{aligned} P_k^c(t)u &:= \sum_{j=1}^{\infty} p_j^{c,k}(t) \langle u, e_j \rangle, \\ M_k^c(t)u &:= \sum_{j=1}^{\infty} m_j^{c,k}(t) \langle u, e_j \rangle, \\ R_k^c(t)u &:= \sum_{j=1}^{\infty} r_j^{c,k}(t) \langle u, e_j \rangle. \end{aligned}$$

The process M_k^c is a weakly square-integrable cylindrical Lévy martingale with a diagonal covariance operator Q_k and the stopping times τ^c , defined in (4.1), satisfy $\tau^c(k) \rightarrow \infty$ P -a.s. as $k \rightarrow \infty$.

Proof. We write $M_k^c(t) = X(t) + Y_k^c(t)$ for each $k > 0$ with

$$X(t)u := \sum_{j=1}^{\infty} \sqrt{s_j} W_j(t) \langle u, e_j \rangle, \quad Y_k^c(t)u := \sum_{j=1}^{\infty} \int_{|x| \leq k/c_j} x \tilde{N}_j(t, dx) \langle u, e_j \rangle,$$

for all $u \in U$. Since condition (4.5) implies

$$\mathbb{E} \left[|X(t)u|^2 \right] = \sum_{j=1}^{\infty} |s_j| \langle u, e_j \rangle^2 \leq \|s\|_{\infty} \|u\|^2,$$

we obtain that $X(t): U \rightarrow L^0(\Omega; \mathbb{R})$ is well defined, continuous and weakly square-integrable. We have

$$\mathbb{E} \left[|Y_k^c(t)u|^2 \right] = t \sum_{j=1}^{\infty} \langle u, e_j \rangle^2 \int_{|x| \leq k/c_j} x^2 \rho_j(dx) < \infty$$

by (4.12). Consequently, $Y_k^c(t)$ and thus $M_k^c(t)$ are well defined, continuous and weakly square-integrable. By (4.11), the (deterministic) process P_k^c is well defined. Since $R_k^c = L - M_k^c - P_k^c$ it follows that the series in the definition of R_k^c converges and that $R_k^c(t) : U \rightarrow L^0(\Omega; \mathbb{R})$ is continuous for all $t \geq 0$. \square

Example 4.4. [Two-sided stable process] An often considered example of a process given in (4.3) is for $\ell_j = \sigma_j h_j$, where h_j are identically distributed, symmetric α -stable Lévy processes and $\sigma_j \in \mathbb{R}$; see [24, 25]. In this case, ℓ_j has Lévy measure $\rho_j = \rho \circ m_{\sigma_j}^{-1}$, where $m_{\sigma_j} : \mathbb{R} \rightarrow \mathbb{R}$ is given by $m_{\sigma_j}(x) = \sigma_j x$ and $\rho(dx) = \frac{1}{2}|x|^{-1-\alpha} dx$. By [28, Ex. 4.5], formula (4.3) defines a cylindrical Lévy process if and only if $\sigma = (\sigma_j) \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{R})$. Moreover, L is induced by a classical process if and only if $\sigma \in \ell^\alpha(\mathbb{R})$.

We show that Assumption (A6) is satisfied for the sequence $(c_j) \in \ell^2(\mathbb{R}_+)$ defined by $c_j = |\sigma_j|^{\frac{\alpha}{2-\alpha}}$. Condition (4.11) is trivially satisfied because each h_j has no drift and the Lévy measure is symmetric. Since

$$\int_{|x| \leq \frac{k}{c_j}} x^2 \rho_j(dx) = \sigma_j^2 \int_{|x| \leq \frac{k}{c_j \sigma_j}} x^2 \rho(dx) = \sigma_j^2 \frac{k^{2-\alpha}}{2-\alpha} |c_j \sigma_j|^{\alpha-2} = \frac{k^{2-\alpha}}{2-\alpha},$$

Condition (4.12) is satisfied. Since $(c_j) \in \ell^2(\mathbb{R}_+)$ by its very definition, Remark 4.2 establishes Condition (4.13).

Example 4.5. [One-sided stable process] We choose $\ell_j = \sigma_j h_j$ in (4.3) with $\sigma_j \in \mathbb{R}$ and h_j arbitrary, strictly α -stable Lévy process with $\alpha \in (0, 2)$ and with no negative jumps. Note, that $\alpha \neq 1$, since a 1-stable Lévy process is strictly stable if and only if its Lévy measure is symmetric. The characteristic function of $h_j(1)$ is given by

$$\varphi_{h_j(1)}(x) = \exp\left(-|x|^\alpha \left(1 - i \tan \frac{\pi\alpha}{2} \operatorname{sgn} x\right)\right),$$

see [30, Th. 14.15, Def. 14.16]. It follows that the Lévy process $\sigma_j h_j$ has characteristics $(b_j, 0, \rho_j)$ given by

$$b_j = \sigma_j \frac{1}{c_\alpha(1-\alpha)} \sigma_j |\sigma_j|^{\alpha-1}, \quad \rho_j(dx) = (\rho \circ m_{\sigma_j}^{-1})(dx),$$

where $c_\alpha = -\cos\left(\frac{\alpha\pi}{2}\right) \Gamma(\alpha)$, the function $m_{\sigma_j} : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $m_{\sigma_j}(x) = \sigma_j x$ and $\rho(dx) = \mathbb{1}_{(0,\infty)}(x) \frac{1}{c_\alpha} x^{-1-\alpha} dx$.

We claim that L is a cylindrical Lévy process if and only if $\sigma \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{R})$. Indeed, Condition (4.4) reduces to

$$\sum_{j=1}^{\infty} |\alpha_j| \left| b_j + \int_{1 < |x| \leq 1/|\alpha_j|} x (\rho \circ m_{\sigma_j}^{-1})(dx) \right| = \frac{c}{c_\alpha |1-\alpha|} \sum_{j=1}^{\infty} |\alpha_j \sigma_j|^\alpha,$$

whereas Condition (4.6) reads as

$$\sum_{j=1}^{\infty} \int_{\mathbb{R}} \left(|\alpha_j x|^2 \wedge 1 \right) \rho_j(dx) = \frac{2c}{c_\alpha(2-\alpha)\alpha} \sum_{j=1}^{\infty} |\alpha_j \sigma_j|^\alpha.$$

Assumption (A6) is satisfied with $c_j = |\sigma_j|^{\frac{\alpha}{2-\alpha}}$, since Condition (4.11) can be calculated as

$$\sum_{j=1}^{\infty} \left(b_j + \int_{1 < |x| \leq \frac{k}{c_j}} x \rho_j(dx) \right)^2 = \left(\frac{ck^{1-\alpha}}{c_\alpha(1-\alpha)} \right)^2 \sum_{j=1}^{\infty} |\sigma_j|^{\frac{2\alpha}{2-\alpha}}.$$

Conditions (4.12) and (4.13) follow by the same arguments as in Example 4.4.

Remark 4.6. In both Examples 4.4 and 4.5, Condition (4.13) would not be satisfied for a constant level of truncation of jumps i.e. with $c_j = 1$ for all $j \in \mathbb{N}$. By introducing the weights (c_j) we compensate the fact that the cylindrical distribution of L is not tight, i.e. its mass of the span of the higher nodes decays too slowly.

Example 4.7. [One-sided regularly varying tails] Recall that a measure μ concentrated on $(0, \infty)$ is said to have regularly varying tails with index α if

$$\lim_{x \rightarrow \infty} \frac{\mu(\lambda x, \infty)}{\mu(x, \infty)} = \lambda^{-\alpha} \quad \text{for all } \lambda > 0;$$

see [3, 8]. We choose $\ell_j = \sigma_j h_j$ in (4.3) with a sequence of independent and identically distributed Lévy processes h_j of regularly varying tails of index $\alpha \in (0, 1) \cup (1, 2)$. For simplifying the calculations, we assume that the characteristic function of $h_j(1)$ is given by

$$\varphi_{h_j(1)}(x) = \begin{cases} \exp \left(\int_0^\infty (e^{ixy} - 1 - ixy \mathbb{1}_{B_{\mathbb{R}}}(y)) \rho(dy) + ixb \right), & \text{if } \alpha \in (0, 1), \\ \exp \left(\int_0^\infty (e^{ixy} - 1 - ixy) \rho(dy) \right), & \text{if } \alpha \in (1, 2), \end{cases}$$

for a constant $b \in \mathbb{R}$. The Lévy measure ρ of h_j has regularly varying tails according to [7].

We show that if $(\sigma_j) \in \ell^{\frac{2\delta}{2-\delta}}(\mathbb{R})$ for some $\delta < \alpha$, then (4.3) defines a cylindrical Lévy process. For this purpose, we define

$$V_\delta(x) := \int_x^\infty y^\delta \mathbb{1}_{(1, \infty)}(y) \rho(dy), \quad U_2(x) := \int_1^x y^2 \rho(dy), \quad \text{for } x > 0.$$

It follows from [29, Prop. 4.2.1] that $V_\delta(0) < \infty$ and $U_2(\infty) = \infty$. Theorem VII.9.2 in [8] implies

$$\lim_{x \rightarrow \infty} \frac{x^{2-\delta} V_\delta(x)}{U_2(x)} = \frac{2-\alpha}{\alpha-\delta} =: c,$$

and therefore there exists $M > 0$ such that

$$U_2(x) \leq \frac{2x^{2-\delta}V_\delta(x)}{c} \quad \text{for all } x \geq M.$$

Since both (α_j) and (σ_j) tend to 0 we can assume without loss of generality that $\frac{1}{|\alpha_j\sigma_j|} > M$ for all $j \in \mathbb{N}$. For verifying Condition (4.6) we obtain

$$\begin{aligned} & \sum_{j=1}^{\infty} \left(\alpha_j^2 \sigma_j^2 \int_{0 \leq x \leq 1} x^2 \rho(dx) + \alpha_j^2 \sigma_j^2 \int_{1 < x \leq \frac{1}{|\alpha_j \sigma_j|}} x^2 \rho(dx) \right) \\ &= \sum_{j=1}^{\infty} \alpha_j^2 \sigma_j^2 \int_{0 \leq x \leq 1} x^2 \rho(dx) + \sum_{j=1}^{\infty} \alpha_j^2 \sigma_j^2 U_2 \left(\frac{1}{|\alpha_j \sigma_j|} \right) \\ &\leq \sum_{j=1}^{\infty} \alpha_j^2 \sigma_j^2 \int_{0 \leq x \leq 1} x^2 \rho(dx) + \frac{2}{c} \sum_{j=1}^{\infty} |\alpha_j \sigma_j|^\delta V_\delta \left(\frac{1}{|\alpha_j \sigma_j|} \right). \end{aligned}$$

Both sums are finite because of the summability assumptions on α and σ . Similarly, we derive that

$$\sum_{j=1}^{\infty} \rho \left(\left[\frac{1}{|\alpha_j \sigma_j|}, \infty \right) \right) < \infty,$$

which shows Condition (4.6).

Similarly to the stable case in Example 4.5, the sequence (c_j) satisfying (A6) can be defined by $c_j = |\sigma_j|^{\frac{\alpha}{2-\alpha}}$.

Note that the conclusion in this example is not optimal in the case of α -stable noise. For, in Example 4.5 we can choose $\sigma \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{R})$ whereas here we have to choose $\sigma \in \ell^{\frac{2\delta}{2-\delta}}(\mathbb{R})$ for $\delta < \alpha$.

The integration theory developed in [27] and improved in Section 3 relies on finite weak moments of the cylindrical Lévy process. In the following, we extend this stochastic integral to the class of cylindrical Lévy processes of the form (4.3) under Assumption (A6) without requiring finite weak moments. For this purpose, by fixing a sequence $c \in \ell^\infty(\mathbb{R}_+)$ such that Assumption (A6) is satisfied and by using the notation (4.8)–(4.10) we define for each $k > 0$:

$$L_k^c(t)u := \sum_{j=1}^{\infty} \left(p_j^{c,k}(t) + m_j^{c,k}(t) \right) \langle u, e_j \rangle, \quad t \geq 0, u \in U.$$

Lemma 4.3 yields that $L_k^c = P_k^c + M_k^c$ is a square-integrable cylindrical Lévy process. Let Q_k denote the covariance operator of M_k^c and \mathcal{H}_k be the corresponding reproducing kernel Hilbert space, where we suppress the dependence on the sequence c in the

notation for Q_k and \mathcal{H}_k . At the same time, we extend the class of integrands by the usual localisation arguments. For this purpose, we define the class Λ_{loc} as

$$\Lambda_{\text{loc}} := \left\{ \Psi : [0, T] \times \Omega \rightarrow L_{\text{HS}}(\mathcal{H}_k, H) : \Psi \text{ is predictable and } \int_0^T \left\| \Psi(t) Q_k^{1/2} \right\|_{L_{\text{HS}}(U, H)}^2 dt < \infty \quad P\text{-a.s. for all } k \in \mathbb{N} \right\}.$$

Theorem 4.8. *Assume that L is a cylindrical Lévy process of the form (4.3) satisfying (A6) for a sequence $c \in \ell^\infty(\mathbb{R}_+)$ and let Ψ be in Λ_{loc} . Then there exists an increasing sequence of stopping times $(\varrho(k))$ with $\varrho(k) \rightarrow \infty$ P -a.s. as $k \rightarrow \infty$ such that $\Psi(\cdot) \mathbb{1}_{[0, \varrho(k)]}(\cdot) \in \Lambda$ for each $k \in \mathbb{N}$ and*

$$\left(\int_0^t \Psi(s) \mathbb{1}_{\{s \leq \varrho(k)\}} dL_k^c(s) : t \in [0, T] \right)_{k \in \mathbb{N}}$$

is a Cauchy sequence in the topology of uniform convergence in probability and its limit is independent of the sequence c satisfying Assumption (A6).

Theorem 4.8 enables us to define for each $\Psi \in \Lambda_{\text{loc}}$ the stochastic integrals

$$\int_0^\cdot \Psi(s) dL(s) := \lim_{k \rightarrow \infty} \int_0^\cdot \Psi(s) \mathbb{1}_{[0, \varrho(k)]}(s) dL_k^c(s),$$

where the limit is taken in the topology of uniform convergence in probability. Note, that although in [13] a stochastic integration theory is developed for a large class of integrands with respect to arbitrary cylindrical Lévy processes, it does not cover the case of only predictable integrands.

Proof. Define the stopping times

$$\tilde{\tau}(k, n) := \inf \left\{ t \in [0, T] : \int_0^t \left\| \Psi(s) Q_k^{1/2} \right\|_{L_{\text{HS}}(U, H)}^2 ds > n \right\},$$

where we take the convention that $\inf \emptyset = +\infty$. Since $\Psi \in \Lambda_{\text{loc}}$, for P -almost every $\omega \in \Omega$ there exists $n(\omega) \in \mathbb{N}$ such that $\int_0^t \left\| \Psi(s)(\omega) Q_k^{1/2} \right\|_{L_{\text{HS}}(U, H)}^2 ds \leq n(\omega)$ for all $t \in [0, T]$, and thus for each $k \in \mathbb{N}$ we have $\tilde{\tau}(k, n)(\omega) = \infty$ for every $n > n(\omega)$. Therefore, we conclude $P(\cup_{n=1}^\infty \{\tilde{\tau}(k, n) = \infty\}) = 1$ for each $k \in \mathbb{N}$. This guarantees that for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $P(\tilde{\tau}(k, n_k) < \infty) \leq \frac{1}{2^k}$. Since $\frac{1}{2^k}$ is summable, it follows by the Borel–Cantelli Lemma that

$$P \left(\limsup_{k \rightarrow \infty} \{\tilde{\tau}(k, n_k) < \infty\} \right) = 0.$$

Consequently, the stopping times $\varrho^c(k) := \tau^c(k) \wedge \tilde{\tau}(k, n_k)$ also converge to $+\infty$ a.s. by Lemma 4.3. Note that if $T < \varrho^c(k)$, then $L_k^c = L_n^c$ on $[0, T]$ and

$$\int_0^t \Psi(s) \mathbb{1}_{\{s \leq \varrho^c(k)\}} dL_k^c(s) = \int_0^t \Psi(s) \mathbb{1}_{\{s \leq \varrho^c(n)\}} dL_n^c(s)$$

for all $t \in [0, T]$. Consequently, we obtain for each $k \leq n$ and $\varepsilon > 0$ that

$$\begin{aligned} & P \left(\sup_{t \in [0, T]} \left\| \int_0^t \Psi(s) \mathbb{1}_{\{s \leq \varrho^c(k)\}} dL_k^c(s) - \int_0^t \Psi(s) \mathbb{1}_{\{s \leq \varrho^c(n)\}} dL_n^c(s) \right\|_H \geq \varepsilon \right) \\ & \leq P \left(\int_0^t \Psi(s) \mathbb{1}_{\{s \leq \varrho^c(k)\}} dL_k^c(s) \neq \int_0^t \Psi(s) \mathbb{1}_{\{s \leq \varrho^c(n)\}} dL_n^c(s) \text{ for some } t \in [0, T] \right) \\ & \leq P(T \geq \varrho^c(k)) \rightarrow 0 \quad \text{as } n, k \rightarrow \infty, \end{aligned}$$

which establishes the claimed convergence.

The limit of the Cauchy sequence does not depend on the choice of the sequence c satisfying (A6) because if d is another sequence satisfying (A6), then $L_k^c = L_n^d$ for all $t \in [0, T]$ on $\{T < \tau^c(k) \wedge \tau^d(n)\}$ and

$$\int_0^t \Psi(s) \mathbb{1}_{\{s \leq \tau^c(k)\}} dL_k^c(s) = \int_0^t \Psi(s) \mathbb{1}_{\{s \leq \tau^d(n)\}} dL_n^d(s),$$

which completes the proof. \square

5 Existence of a solution for the orthogonal noise

Existence of a cylindrical Lévy process of the form (4.3) strongly depends on the interplay between the drift part b_j and the Lévy measure ρ_j of the real valued Lévy process with characteristics (b_j, s_j, ρ_j) , see condition (4.4). For this reason, we consider the general case of a cylindrical Lévy process with a possible non-zero drift part. Naturally, we will tackle this part by moving it to the drift part of the equation under consideration. For this purpose, recall the decomposition $L(t) = P_k^c(t) + M_k^c(t) + R_k^c(t)$ of the cylindrical Lévy process L for each $k > 0$ derived in Lemma 4.3 under assumption (A6) satisfied for a sequence $c \in \ell^\infty(\mathbb{R}_+)$. Let Q_k denote the covariance operator of M_k^c where we suppress the dependency on the sequence c . Furthermore, instead of the standard coercivity and monotonicity requirements, we introduce assumptions for each truncation level $k \in \mathbb{N}$. Assumptions of this form were introduced in Peszat and Zabczyk [21, Sec. 9.7] in the semigroup approach. Assume that there are constants $\alpha_k, \lambda_k, \beta_k > 0$ such that

(A1') (coercivity) For every $k \in \mathbb{N}$ and $v \in V$ we have

$$2_{V^*} \langle F(v) + P_k^c(1)G^*(v), v \rangle_V + \|G(v)Q_k^{1/2}\|_{L_{\text{HS}}(U, H)}^2 + \alpha_k \|v\|_V^2 \leq \lambda_k \|v\|_H^2 + \beta_k;$$

(A2') (monotonicity) For every $k \in \mathbb{N}$ and $v_1, v_2 \in V$ we have

$$2_{V^*} \langle F(v_1) - F(v_2) + P_k^c(1)(G^*(v_1) - G^*(v_2)), v_1 - v_2 \rangle_V \\ + \|(G(v_1) - G(v_2))Q_k^{1/2}\|_{L_{\text{HS}}(U, H)}^2 \leq \lambda_k \|v_1 - v_2\|_H.$$

(A3') (linear growth) $\|F(v) + P_k^c(1)G^*(v)\|_{V^*} \leq c_k(1 + \|v\|_V)$ for all $v \in V$;

(A4') (hemicontinuity) the mapping

$$\mathbb{R} \ni s \mapsto {}_{V^*} \langle F(v_1 + sv_2) + P_k^c(1)G^*(v_1 + sv_2), v_3 \rangle_V$$

is continuous for all $v_1, v_2, v_3 \in V$.

Note, the term $P_k^c(1)$ is due to the fact that we allow general cylindrical Lévy processes with a possible non-zero drift part. Naturally, such a non-zero drift part of the noise contributes to the dynamic of the equation (3.2) together with the diffusion coefficient G as an addition to the drift coefficient F . For example, for a cylindrical Lévy process with symmetric cylindrical Lévy measure and drift $p = 0$ in (2.1), the term $P_k^c(1)$ disappears and these assumptions simplify to the more familiar conditions (A1)–(A4) but with Q_k instead of Q .

Theorem 5.1. *Assume that L is a cylindrical Lévy process of the form (4.3) satisfying (A6) on page 11 for a sequence $c \in \ell^\infty(\mathbb{R}_+)$. If the coefficients F and G satisfy (A1')–(A4'), then equation (3.2) with an \mathcal{F}_0 -measurable initial condition $X(0) = X_0$ has a pathwise unique variational solution (X, \bar{X}) .*

Proof. We reduce the case of the general initial condition to the square integrable one as in [1, Th. 6.2.3]. For $k \in \mathbb{N}$ let $\Omega_k = \{\|X_0\| \leq k\}$ and $X_0^k = X_0 \mathbb{1}_{\Omega_k}$. Using the decomposition $L(t) = P_k^c(t) + M_k^c(t) + R_k^c(t)$, Lemma 4.3 guarantees that M_k^c is a weakly square-integrable cylindrical Lévy martingale with diagonal covariance, and thus according to Theorem 3.3 there exists a unique variational solution (X_k^c, \bar{X}_k^c) of

$$dX(t) = (F(X(t)) + P_k^c(1)G^*(X(t)))dt + G(X(t))dM_k^c(t),$$

with the initial condition $X(0) = X_0^k$.

Step 1. We first show that for each $k \leq n$ we have $X_k^c(t) = X_n^c(t)$ P -a.s. on $\{t < \tau^c(k)\} \cap \Omega_k$.

Define a cylindrical Lévy process $Y_{k,n}^c$ by

$$Y_{k,n}^c(t)u := R_k^c(t)u - R_n^c(t)u = \sum_{j=1}^{\infty} \left(\int_{k/c_j < |x| \leq n/c_j} x N_j(t, dx) \right) \langle u, e_j \rangle$$

for all $t \geq 0$ and $u \in U$. The cylindrical martingale M_n^c can be rewritten as

$$\begin{aligned}
& M_n^c(t)u \\
&= \sum_{j=1}^{\infty} \left(m_j^{c,k}(t) + \int_{k/c_j < |x| \leq n/c_j} x \tilde{N}_j(t, dx) \right) \langle u, e_j \rangle \\
&= \sum_{j=1}^{\infty} \left(m_j^{c,k}(t) + \int_{k/c_j < |x| \leq n/c_j} x N_j(t, dx) - \int_{k/c_j < |x| \leq n/c_j} x \rho_j(t, dx) \right) \langle u, e_j \rangle \\
&= M_k^c(t)u + Y_{k,n}^c(t)u - (P_n^c(1)u - P_k^c(1)u)t.
\end{aligned}$$

Applying this we get

$$\begin{aligned}
X_k^c(t) - X_n^c(t) &= -X_0 \mathbb{1}_{\Omega_n \setminus \Omega_k} + \int_0^t (F(\bar{X}_k^c(s)) - F(\bar{X}_n^c(s))) ds \\
&\quad + \int_0^t P_k^c(1) (G^*(\bar{X}_k^c(s)) - G^*(\bar{X}_n^c(s))) ds \\
&\quad + \int_0^t (G(\bar{X}_k^c(s)) - G(\bar{X}_n^c(s))) dM_k^c(s) \\
&\quad - \int_0^t G(\bar{X}_n^c(s)) dY_{k,n}^c(s).
\end{aligned} \tag{5.1}$$

By introducing the new notation

$$\begin{aligned}
A(t) &:= X_k^c(t) - X_n^c(t) + \int_0^t G(\bar{X}_n^c(s)) dY_{k,n}^c(s), \\
I(t) &:= \int_0^t (G(\bar{X}_k^c(s)) - G(\bar{X}_n^c(s))) dM_k^c(s),
\end{aligned}$$

equation (5.1) can be rewritten as

$$\begin{aligned}
A(t) &= -X_0 \mathbb{1}_{\Omega_n \setminus \Omega_k} + \int_0^t (F(\bar{X}_k^c(s)) - F(\bar{X}_n^c(s))) ds \\
&\quad + \int_0^t P_k^c(1) (G^*(\bar{X}_k^c(s)) - G^*(\bar{X}_n^c(s))) ds + I(t).
\end{aligned} \tag{5.2}$$

Note that on $\{t < \tau^c(k)\}$ we have $A(t) = X_k^c(t) - X_n^c(t)$. We apply Theorem 1 in [11] with $v(t) = \bar{X}_k^c(t) - \bar{X}_n^c(t)$ and $h(t) = I(t)$ on the set $\{t < \tau^c(k)\}$. It follows that there exists an H -valued, càdlàg process Z , which is equal to $A = X_k^c - X_n^c$ Leb $\otimes P$ -almost everywhere on $\Upsilon := \{(t, \omega) \in [0, T] \times \Omega : t < \tau^c(k)(\omega)\}$ and such that the Itô formula for the square of the norm holds:

$$\|Z(t)\|_H^2$$

$$\begin{aligned}
&= \|X_0\|_H^2 \mathbb{1}_{\Omega_n \setminus \Omega_k} + 2 \int_0^t V^* \langle F(\bar{X}_k^c(s)) - F(\bar{X}_n^c(s)), \bar{X}_k^c(s) - \bar{X}_n^c(s) \rangle_V ds \\
&\quad + 2 \int_0^t V^* \langle P_k^c(1) (G^*(\bar{X}_k^c(s)) - G^*(\bar{X}_n^c(s))), \bar{X}_k^c(s) - \bar{X}_n^c(s) \rangle_V ds \\
&\quad + \int_0^t (X_k^c(s-) - X_n^c(s-)) dI(s) + [I, I](t).
\end{aligned} \tag{5.3}$$

We show that $X_k^c - X_n^c$ and Z are indistinguishable on Υ . We have that

$$\mathbb{E} \left[\int_0^{\tau^c(k)} \mathbb{1}_{\{Z(t) \neq (X_k^c - X_n^c)(t)\}} dt \right] = 0.$$

This implies that there exists $\Omega_1 \subset \Omega$ with $P(\Omega_1) = 1$ such that for all $\omega \in \Omega_1$

$$\int_0^{\tau^c(k)(\omega)} \mathbb{1}_{\{Z(t, \omega) \neq (X_k^c - X_n^c)(t, \omega)\}} dt = 0.$$

We obtain that for every $\omega \in \Omega_1$ there is a subset $A_\omega \subset [0, \tau^c(k)(\omega))$ with $\text{Leb}(A_\omega) = \tau^c(k)(\omega)$ and such that $Z(t, \omega) = (X_k^c - X_n^c)(t, \omega)$ for all $t \in A_\omega$. Note that $X_k^c - X_n^c$ is a càdlàg process in V^* . Fix $\omega \in \Omega_1$ and $t \in [0, \tau^c(k)(\omega))$. Let $(t_m) \subset A_\omega$ be a sequence decreasing to t . Since $Z(t_m, \omega) = (X_k^c - X_n^c)(t_m, \omega)$ for all $m \in \mathbb{N}$ we obtain that $Z(t, \omega) = (X_k^c - X_n^c)(t, \omega)$ for all $\omega \in \Omega_1$ and $t \in [0, \tau^c(k)(\omega))$ i.e. Z and $X_k^c - X_n^c$ are indistinguishable on Υ .

Thus, in what follows, we can assume that for $t < \tau^c(k)$ the process $X_k^c - X_n^c$ is H -valued and càdlàg. Let

$$J(t) := \int_0^t (X_k^c(s-) - X_n^c(s-)) dI(s).$$

We show that

$$\begin{aligned}
&\|A(t \wedge \tau^c(k))\|_H^2 \\
&= \|X_0\|_H^2 \mathbb{1}_{\Omega_n \setminus \Omega_k} + 2 \int_0^{t \wedge \tau^c(k)} V^* \langle F(\bar{X}_k^c(s)) - F(\bar{X}_n^c(s)), \bar{X}_k^c(s) - \bar{X}_n^c(s) \rangle_V ds \\
&\quad + 2 \int_0^{t \wedge \tau^c(k)} V^* \langle P_k^c(1) (G^*(\bar{X}_k^c(s)) - G^*(\bar{X}_n^c(s))), \bar{X}_k^c(s) - \bar{X}_n^c(s) \rangle_V ds \\
&\quad + J(t \wedge \tau^c(k)) + [I, I](t \wedge \tau^c(k)).
\end{aligned} \tag{5.4}$$

It follows from (5.3) by taking the left limit at $t \wedge \tau^c(k)$ that

$$\|X_k^c((t \wedge \tau^c(k))-) - X_n^c((t \wedge \tau^c(k))-)\|_H^2$$

$$\begin{aligned}
&= \|X_0\|_H^2 \mathbb{1}_{\Omega_n \setminus \Omega_k} + 2 \int_0^{t \wedge \tau^c(k)} V^* \langle F(\bar{X}_k^c(s)) - F(\bar{X}_n^c(s)), \bar{X}_k^c(s) - \bar{X}_n^c(s) \rangle_V ds \\
&\quad + 2 \int_0^{t \wedge \tau^c(k)} V^* \langle P_k^c(1)(G^*(\bar{X}_k^c(s)) - G^*(\bar{X}_n^c(s))), \bar{X}_k^c(s) - \bar{X}_n^c(s) \rangle_V ds \\
&\quad + J((t \wedge \tau^c(k)) -) + [I, I]((t \wedge \tau^c(k)) -). \tag{5.5}
\end{aligned}$$

By definition of A , $A(s) = X_k^c(s) - X_n^c(s)$ for $s < \tau^c(k)$. Taking the limits as $s \nearrow t \wedge \tau^c(k)$ we get that $A((t \wedge \tau^c(k)) -) = X_k^c((t \wedge \tau^c(k)) -) - X_n^c((t \wedge \tau^c(k)) -)$. Since the only discontinuous processes in (5.2) are A and I , it follows that $\Delta A(t \wedge \tau^c(k)) = \Delta I(t \wedge \tau^c(k))$. Thus

$$\begin{aligned}
\|A(t \wedge \tau^c(k))\|_H^2 &= \|A((t \wedge \tau^c(k)) -) + \Delta A(t \wedge \tau^c(k))\|_H^2 \\
&= \|X_k^c((t \wedge \tau^c(k)) -) - X_n^c((t \wedge \tau^c(k)) -) + \Delta I(t \wedge \tau^c(k))\|_H^2 \\
&= \|X_k^c((t \wedge \tau^c(k)) -) + X_n^c((t \wedge \tau^c(k)) -)\|_H^2 + \|\Delta I(t \wedge \tau^c(k))\|_H^2 \\
&\quad + \langle \Delta I(t \wedge \tau^c(k)), X_k^c((t \wedge \tau^c(k)) -) - X_n^c((t \wedge \tau^c(k)) -) \rangle_H.
\end{aligned}$$

Applying (5.5) we obtain

$$\begin{aligned}
&\|A(t \wedge \tau^c(k))\|_H^2 \\
&= \|X_0\|_H^2 \mathbb{1}_{\Omega_n \setminus \Omega_k} + \int_0^{t \wedge \tau^c(k)} V^* \langle F(\bar{X}_k^c(s)) - F(\bar{X}_n^c(s)), \bar{X}_k^c(s) - \bar{X}_n^c(s) \rangle_V ds \\
&\quad + \int_0^{t \wedge \tau^c(k)} V^* \langle P_k^c(1)(G^*(\bar{X}_k^c(s)) - G^*(\bar{X}_n^c(s))), \bar{X}_k^c(s) - \bar{X}_n^c(s) \rangle_V ds \\
&\quad + J((t \wedge \tau^c(k)) -) + [I, I]((t \wedge \tau^c(k)) -) \\
&\quad + \langle \Delta I(t \wedge \tau^c(k)), X_k^c((t \wedge \tau^c(k)) -) - X_n^c((t \wedge \tau^c(k)) -) \rangle_H + \|\Delta I(t \wedge \tau^c(k))\|_H^2 \tag{5.6}
\end{aligned}$$

The jump of the stochastic integral J at $t \wedge \tau^c(k)$ equals to

$$\langle X_k^c((t \wedge \tau^c(k)) -) - X_n^c((t \wedge \tau^c(k)) -), \Delta I(t \wedge \tau^c(k)) \rangle_H,$$

see [20, Prop. 24.3 and Sec. 26.4]. Thus in (5.6) we can simplify

$$\begin{aligned}
&J((t \wedge \tau^c(k)) -) + \langle \Delta I(t \wedge \tau^c(k)), X_k^c((t \wedge \tau^c(k)) -) - X_n^c((t \wedge \tau^c(k)) -) \rangle_H \\
&= J((t \wedge \tau^c(k)) -) + \Delta J(t \wedge \tau^c(k)) \\
&= J(t \wedge \tau^c(k)). \tag{5.7}
\end{aligned}$$

Similarly, we obtain that the jump of the quadratic variation of I at $t \wedge \tau^c(k)$ equals $\|\Delta I(t \wedge \tau^c(k))\|_H^2$, see [20, Th. 20.5(4)], and we can simplify in (5.6):

$$[I, I]((t \wedge \tau^c(k)) -) + \|\Delta I(t \wedge \tau^c(k))\|_H^2 = [I, I]((t \wedge \tau^c(k)) -) + \Delta[I, I](t \wedge \tau^c(k))$$

$$= [I, I](t \wedge \tau^c(k)). \quad (5.8)$$

Applying (5.7) and (5.8) in (5.6) finishes the proof of (5.4).

We multiply both sides of (5.4) by $\mathbb{1}_{\Omega_k}$ and take expectation. For the term involving the quadratic variation, we use $\mathbb{E}[[I, I](t \wedge \tau^c(k))] = \mathbb{E}[\langle I, I \rangle(t \wedge \tau^c(k))]$ and Theorem 3.1(ii). Recall for the following that the martingale property is invariant under multiplication by $\mathbb{1}_{\Omega_k}$, since Ω_k is \mathcal{F}_0 -measurable. Thus $\mathbb{E}[J(t \wedge \tau^c(k))\mathbb{1}_{\Omega_k}] = 0$. We obtain

$$\begin{aligned} & \mathbb{E} \left[\|A(t \wedge \tau^c(k))\|_H^2 \mathbb{1}_{\Omega_k} \right] \\ &= \mathbb{E} \left[2 \int_0^{t \wedge \tau^c(k)} V^* \langle F(\bar{X}_k^c(s)) - F(\bar{X}_n^c(s)), \bar{X}_k^c(s) - \bar{X}_n^c(s) \rangle_V ds \mathbb{1}_{\Omega_k} \right] \\ &+ \mathbb{E} \left[2 \int_0^{t \wedge \tau^c(k)} V^* \langle P_k^c(1)(G^*(\bar{X}_k^c(s)) - G^*(\bar{X}_n^c(s))), \bar{X}_k^c(s) - \bar{X}_n^c(s) \rangle_V ds \mathbb{1}_{\Omega_k} \right] \\ &+ \mathbb{E} \left[\int_0^{t \wedge \tau^c(k)} \| (G(\bar{X}_k^c(s)) - G(\bar{X}_n^c(s))) Q_k^{1/2} \|_{L_{HS}(U, H)}^2 ds \mathbb{1}_{\Omega_k} \right]. \end{aligned}$$

The monotonicity assumption (A2') implies that

$$\begin{aligned} \mathbb{E} \left[\|A(t \wedge \tau^c(k))\|_H^2 \mathbb{1}_{\Omega_k} \right] &\leq \lambda_k \mathbb{E} \left[\int_0^{t \wedge \tau^c(k)} \|X_k^c(s) - X_n^c(s)\|_H^2 ds \mathbb{1}_{\Omega_k} \right] \\ &\leq \lambda_k \mathbb{E} \left[\int_0^t \|A(s \wedge \tau^c(k))\|_H^2 ds \mathbb{1}_{\Omega_k} \right]. \end{aligned}$$

It follows by Gronwall's inequality that

$$\mathbb{E} \left[\left\| X_k^c(t \wedge \tau^c(k)) - X_n^c(t \wedge \tau^c(k)) + \int_0^{t \wedge \tau^c(k)} G(\bar{X}_n^c(s)) dY_{k,n}^c(s) \right\|_H^2 \mathbb{1}_{\Omega_k} \right] = 0.$$

Thus

$$\left(X_k^c(t \wedge \tau^c(k)) - X_n^c(t \wedge \tau^c(k)) + \int_0^{t \wedge \tau^c(k)} G(\bar{X}_n^c(s)) dY_{k,n}^c(s) \right) \mathbb{1}_{\Omega_k} = 0 \quad \text{a.s.}$$

In particular we obtain that

$$X_k^c(t) - X_n^c(t) = 0 \text{ a.s. on } \{t < \tau^c(k)\} \cap \Omega_k.$$

Step 2. The first part enables us to define

$$X := X_k^c \quad \text{and} \quad \bar{X} := \bar{X}_k^c \quad \text{on } \{t < \tau^c(k)\} \cap \Omega_k. \quad (5.9)$$

This definition does not depend on the choice of the sequence c : for, if d is another sequence in $\ell^\infty(\mathbb{R})$ satisfying (A6), then one can show similarly as in Step 1, that $X_k^c = X_n^d$ on $\{t < \tau^c(k) \wedge \tau^d(n)\}$. Moreover, since X_k^c and \bar{X}_k^c are equal almost everywhere on $\{(\omega, t) \in \Omega_k \times [0, T] : t < \tau^c(k)(\omega)\}$, it follows by taking $k \rightarrow \infty$ that X and \bar{X} are equal almost everywhere on $\Omega \times [0, T]$.

Step 3. We show that (X, \bar{X}) defined in (5.9) satisfies (3.3). Note that

$$\begin{aligned} X(t) \mathbb{1}_{\{t < \tau^c(k)\} \cap \Omega_k} &= X_k^c(t) \mathbb{1}_{\{t < \tau^c(k)\} \cap \Omega_k} \\ &= X_0 \mathbb{1}_{\Omega_k} + \mathbb{1}_{\{t < \tau^c(k)\} \cap \Omega_k} \int_0^t F(\bar{X}_k^c(s)) \, ds + \mathbb{1}_{\{t < \tau^c(k)\} \cap \Omega_k} \int_0^t G(\bar{X}_k^c(s)) \, dL_k^c(s). \end{aligned} \quad (5.10)$$

From the very definition (5.9) it follows

$$\begin{aligned} \lim_{k \rightarrow \infty} \mathbb{1}_{\{t < \tau^c(k)\} \cap \Omega_k} \int_0^t F(\bar{X}_k^c(s)) \, ds &= \lim_{k \rightarrow \infty} \mathbb{1}_{\{t < \tau^c(k)\} \cap \Omega_k} \int_0^t F(\bar{X}(s)) \, ds \\ &= \int_0^t F(\bar{X}(s)) \, ds. \end{aligned} \quad (5.11)$$

The last term in (5.10) can be rewritten as

$$\mathbb{1}_{\{t < \tau^c(k)\} \cap \Omega_k} \int_0^t G(\bar{X}_k^c(s)) \, dL_k^c(s) = \mathbb{1}_{\{t < \tau^c(k)\} \cap \Omega_k} \int_0^{t \wedge \tau^c(k)} G(\bar{X}_k^c(s)) \, dL_k^c(s).$$

From Theorem 3.1(iii) and the definition of the stochastic integral with respect to L after Theorem 4.8, it follows that

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_0^{t \wedge \tau^c(k)} G(\bar{X}_k^c(s)) \, dL_k^c(s) &= \lim_{k \rightarrow \infty} \int_0^t G(\bar{X}_k^c(s)) \mathbb{1}_{\{s \leq \tau^c(k)\}} \, dL_k^c(s) \\ &= \lim_{k \rightarrow \infty} \int_0^t G(\bar{X}(s)) \mathbb{1}_{\{s \leq \tau^c(k)\}} \, dL_k^c(s) \\ &= \int_0^t G(\bar{X}(s)) \, dL(s). \end{aligned} \quad (5.12)$$

By taking the limit $k \rightarrow \infty$ in (5.10), equalities (5.11) and (5.12) show

$$X(t) = X_0 + \int_0^t F(\bar{X}(s)) \, ds + \int_0^t G(\bar{X}(s)) \, dL(s),$$

which finishes the proof of the theorem. \square

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